IP326. Solution to Final Exam

1. Given $\dot{x}(t) = g(x(t))\,\theta(t)$, and ...(1)

$$P(x,t) = \langle \delta(x - x(t)) \rangle \qquad \ldots(2)$$

Differentiate (2) w.r.t. t:

$$\frac{\partial P(x,t)}{\partial t} = \left\langle \frac{\partial}{\partial t} \delta(x - x(t)) \right\rangle$$

$$= \left\langle \frac{\partial}{\partial x(t)} \delta(x - x(t))\, \dot{x}(t) \right\rangle$$

$$= -\frac{\partial}{\partial x} \left\langle \delta(x - x(t))\, g(x(t))\, \theta(t) \right\rangle$$

$$= -\frac{\partial}{\partial x} g(x) \left\langle \delta(x - x(t))\, \theta(t) \right\rangle \qquad \ldots(3)$$

From Novikov's theorem, $\langle \delta(x-x(t))\theta(t)\rangle = \int_0^t dt' \langle \theta(t)\theta(t')\rangle \left\langle \frac{\delta}{\delta\theta(t')} \delta(x - x(t)) \right\rangle$ ...(4)

$$\left\langle \frac{\delta}{\delta\theta(t')} \delta(x - x(t)) \right\rangle = \left\langle \frac{\delta}{\delta x(t)} \delta(x - x(t)) \frac{\delta x(t)}{\delta\theta(t')} \right\rangle$$

$$= -\frac{\partial}{\partial x} \left\langle \delta(x - x(t)) \frac{\delta x(t)}{\delta\theta(t')} \right\rangle \qquad \ldots(5)$$

From (1), $\frac{d}{dt} \frac{\delta x(t)}{\delta\theta(t')} = g'(x(t))\,\theta(t) . \frac{\delta x(t)}{\delta\theta(t')} + g(x(t))\,\delta(t-t')$ ...(6)

where $g'(x(t))$ denotes differentiation w.r.t. $x(t)$

Eq.(6) is a linear first order ODE in $\delta x(t)/\delta\theta(t')$ that can be solved using integrating factors. In this case the integrating factor is

$$I(t) = I(0) \exp\left[-\int_0^t ds\, g'(x(s))\,\theta(s) \right] \qquad \ldots(7)$$

Assuming $\delta x(0)/\delta\theta(t') = 0$, the solution of (6) is

$$\frac{\delta x(t)}{\delta\theta(t')} I(0) \exp\left[-\int_0^t ds\, g'(x(s))\,\theta(s)\right] = \int_0^t ds'\, I(0) \exp\left[-\int_0^{s'} ds\, g'(x(s))\,\theta(s)\right] . g(x(s'))\,\delta(s'-t')$$

That is, $\frac{\delta x(t)}{\delta\theta(t')} = \int_0^t ds' \exp\left[\int_{s'}^t ds\, g'(x(s))\theta(s)\right] g(x(s'))\,\delta(s'-t')$

$$= g(x(t'))\,H(t-t') \exp\left[\int_{t'}^t ds\, g'(x(s))\,\theta(s)\right] \quad \ldots(8)$$

where ~~H~~ $H(t-t')$ is the Heaviside step function.

Substituting (8) in (5), and the result in (4), we get

$$\langle\delta(x-x(t))\theta(t)\rangle = 2\lambda \int_0^t dt'\, \delta(t-t')\left(-\frac{\partial}{\partial x}\right)\left\langle \delta(x-x(t))\, g(x(t')) \exp\int_{t'}^t ds\, g'(x(s))\theta(s)\right\rangle . H(t-t')$$

$$= -\lambda \frac{\partial}{\partial x}\langle\delta(x-x(t))\,g(x(t))\rangle \qquad (\text{using } H(0) = 1/2)$$

$$= -\lambda \frac{\partial}{\partial x} g(x)\langle\delta(x-x(t))\rangle = -\lambda\frac{\partial}{\partial x} g(x)\,P(x,t) \quad \ldots(9)$$

Substituting (9) in (3),

$$\frac{\partial P(x,t)}{\partial t} = \lambda \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x)\, P(x,t)$$

which is the required Fokker-Planck equation.

a) Given $e^{i\mathcal{L}t} = e^{Qi\mathcal{L}t} + e^{i\mathcal{L}t} S(t)$ $\quad \ldots(1)$

Introduce $P+Q = 1$ into (1)

$$e^{i\mathcal{L}t} = e^{Qi\mathcal{L}t} + e^{i\mathcal{L}t}(P+Q)\,S(t)$$

$$= e^{Qi\alpha t} + e^{i\alpha t} P S(t) + e^{i\alpha t} Q S(t)$$

$$\therefore \quad e^{i\alpha t}\left(1 - Q S(t)\right) = e^{Qi\alpha t} + e^{i\alpha t} P S(t) \qquad \ldots(2)$$

Multiply (2) from the right by $(1 - Q S(t))^{-1}$

$$\therefore \quad e^{i\alpha t} = e^{Qi\alpha t}\left(1 - Q S(t)\right)^{-1} + e^{i\alpha t} P S(t)\left(1 - Q S(t)\right)^{-1} \qquad \ldots(3)$$

(b) By definition, (3) becomes $e^{i\alpha t} = e^{i\alpha t} P T(t) + V(t)$ $\qquad \ldots(4)$

(c) Given $e^{i\alpha t} = e^{i\alpha t} P + e^{i\alpha t} Q$ $\qquad \ldots(5)$

Substitute (4) into (5),

$$e^{i\alpha t} = e^{i\alpha t} P + \left(V(t) + e^{i\alpha t} P T(t)\right) Q \qquad \ldots(6)$$

Multiply (6) from the right by $i\alpha A$,

$$\therefore \quad e^{i\alpha t} i\alpha A = e^{i\alpha t} P i\alpha A + e^{i\alpha t} P T(t) Q i\alpha A + V(t) Q i\alpha A \qquad \ldots(7)$$

But $e^{i\alpha t} i\alpha A = \dfrac{dA(t)}{dt}$

$\therefore$ (7) becomes, $\dfrac{dA(t)}{dt} = e^{i\alpha t} P i\alpha A + e^{i\alpha t} P T(t) Q i\alpha A + V(t) Q i\alpha A$ $\qquad \ldots(8)$

(d) Using $P = |A\rangle\langle A|A\rangle^{-1}\langle A|$ in the ~~1s~~ 1st and 2nd terms on the RHS of (8),

$$\frac{dA(t)}{dt} = e^{i\alpha t} |A\rangle\langle A|A\rangle^{-1}\langle A|i\alpha A\rangle + e^{i\alpha t} |A\rangle\langle A|A\rangle^{-1}\langle A|T(t) Q i\alpha A\rangle$$

$$+ V(t) Q i\alpha A \qquad \ldots(9)$$

By definition, $\Omega = \langle A|A\rangle^{-1} \langle A|\mathcal{L}A\rangle$, $\Phi(t) = -\langle A|A\rangle^{-1}\langle A|\Gamma(t) Q i\mathcal{L}A\rangle$, and $g(t) = V(t) Q i\mathcal{L}A$

Also $e^{i\mathcal{L}t}|A\rangle = A(t)$

$\therefore$ (9) becomes, $\dfrac{dA(t)}{dt} = i\Omega A(t) - \Phi(t) A(t) + g(t)$

which is the convolutionless GLE.

(e) By definition, $g(t) = V(t) Q i\mathcal{L}A = e^{Qi\mathcal{L}t} \dfrac{1}{1 - Q S(t)} Q i\mathcal{L}A$

$$\therefore \langle g(t)|A\rangle = \left\langle e^{Qi\mathcal{L}t} \frac{1}{1-QS(t)} Q i\mathcal{L}A \middle| A\right\rangle$$

$$= \left\langle e^{Qi\mathcal{L}t}\left(1 + QS(t) + QS(t)QS(t) + \cdots\right) Q i\mathcal{L}A \middle| A\right\rangle$$

$$= \left\langle e^{Qi\mathcal{L}t}\left(Q + QS(t)Q + QS(t)QS(t)Q + \cdots\right) i\mathcal{L}A \middle| A\right\rangle$$

$$= \left\langle e^{Qi\mathcal{L}t} Q\left(1 + S(t)Q + S(t)QS(t)Q + \cdots\right) i\mathcal{L}A \middle| A\right\rangle$$

$$= \left\langle e^{Qi\mathcal{L}t} Q \frac{1}{1 - S(t)Q} i\mathcal{L}A \middle| A\right\rangle$$

$$= \left\langle \left(1 + Qi\mathcal{L}t + \frac{1}{2!} Qi\mathcal{L}t\, Qi\mathcal{L}t + \cdots\right) Q \frac{1}{1-S(t)Q} i\mathcal{L}A \middle| A\right\rangle$$

$$= \left\langle Q\left(1 + Qi\mathcal{L}t + \frac{1}{2!} Qi\mathcal{L}t\, Qi\mathcal{L}t + \cdots\right) Q \frac{1}{1-S(t)Q} i\mathcal{L}A \middle| A\right\rangle$$

(by idempotency of Q, i.e., $Q^2 = Q$)

$$= \langle Q e^{Qi\mathcal{L}t} Q \frac{1}{1-S(t)Q} i\mathcal{L}A | A \rangle$$

$$= \langle e^{Qi\mathcal{L}t} Q (1-S(t)Q)^{-1} i\mathcal{L}A | QA \rangle$$ (by Hermitian property of Q)

$= 0$ (because $Q|A\rangle = 0$ by construction)

3. Given $\Omega_0 = \frac{1}{\langle x|x\rangle} \langle x | \mathcal{L} x \rangle$ ...(1)

where $\mathcal{L} = D \exp(\beta U(x)) \frac{\partial}{\partial x} \exp(-\beta U(x)) \frac{\partial}{\partial x}$

By definition

$$\langle x | \mathcal{L} x \rangle = \int dx \, \psi_{eq}(x) \, x \, \mathcal{L} \, x \quad \text{...(2)}$$

where ~~ψ~~ $\psi_{eq}(x)$ is the equilibrium distribution of $x = C e^{-\beta U(x)}$, where C is a normalization constant

$$\therefore \langle x | \mathcal{L} x \rangle = C \int dx \, e^{-\beta U(x)} x \, D \, e^{\beta U(x)} \frac{\partial}{\partial x} e^{-\beta U(x)} \frac{\partial}{\partial x} x$$

$$= CD \int dx \, x \frac{\partial}{\partial x} e^{-\beta U(x)} \quad \text{...(3)}$$

$$= CD \int dx \, x \, e^{-\beta U(x)} (-\beta) \frac{\partial U(x)}{\partial x} = -\beta D \left\langle x \frac{\partial U(x)}{\partial x} \right\rangle$$

Since $\langle x | x \rangle = \langle x^2 \rangle$

$$\Omega_0 = -\frac{D}{k_B T} \frac{1}{\langle x^2 \rangle} \left\langle x \frac{\partial U(x)}{\partial x} \right\rangle$$

Eq (3) can also be treated by partial integration

$$\langle x | \mathcal{L} x \rangle = CD \left[x e^{-\beta U(x)} \Big|_{-\infty}^{\infty} - \int dx \, e^{-\beta U(x)} \right]$$

$$= -D\int dx\, \psi_{eq}(x) = -D$$

$\therefore$ We also have $\Omega_0 = -\dfrac{D}{\langle x^2\rangle}$

b) Given $\hat{K}(s) = \dfrac{1}{\langle x|x\rangle}\langle x|\mathcal{L}Q \dfrac{1}{s-Q\mathcal{L}} Q\mathcal{L}|x\rangle$ ...(1)

Define $|x_1\rangle = Q\mathcal{L}|x\rangle$, $\therefore \langle x_1| = \langle x|\mathcal{L}Q$

$$\therefore \hat{K}(s) = \frac{1}{\langle x|x\rangle}\langle x_1|\frac{1}{s-Q\mathcal{L}}|x_1\rangle$$

$$= \frac{1}{\langle x|x\rangle}\langle x_1|\frac{1}{s-Q\mathcal{L}(Q_1+P_1)}|x_1\rangle \quad ...(2)$$

where

$$P_1 = |x_1\rangle\langle x_1|x_1\rangle^{-1}\langle x_1|$$

$$\hat{K}(s) = \frac{1}{\langle x|x\rangle}\langle x_1|\frac{1}{s-Q\mathcal{L}Q_1 - Q\mathcal{L}P_1}|x_1\rangle$$

$$= \frac{1}{\langle x|x\rangle}\langle x_1|\left(\frac{1}{s-Q\mathcal{L}Q_1} + \frac{1}{s-Q\mathcal{L}Q_1}Q\mathcal{L}P_1\frac{1}{s-Q\mathcal{L}}\right)|x_1\rangle \quad ...(3)$$

(using the operator identity discussed in class)

$$\therefore \hat{K}(s) = \frac{1}{\langle x|x\rangle}\langle x_1|\frac{1}{s-Q\mathcal{L}Q_1}|x_1\rangle + \frac{1}{\langle x|x\rangle}\langle x_1|\frac{1}{s-Q\mathcal{L}Q_1}Q\mathcal{L}|x_1\rangle\langle x_1|x_1\rangle^{-1}$$

$$\cdot\langle x_1|\frac{1}{s-Q\mathcal{L}}|x_1\rangle \quad ...(4)$$

(using the definition of P_1 in the 2nd term on the RHS of (3))

That is, $\hat{K}(s) \equiv \hat{K}_1(s) + \hat{K}_2(s)$

Consider $\hat{K}_1(s) = \frac{1}{\langle x|x\rangle} \langle x_1 | \frac{1}{s - Q\mathcal{L}Q_1} | x_1 \rangle$

$$= \frac{1}{\langle x|x\rangle} \langle x_1 | \frac{1}{s} \left(1 + \frac{1}{s} Q\mathcal{L}Q_1 + \frac{1}{s^2} Q\mathcal{L}Q_1 Q\mathcal{L}Q_1 + \cdots \right) | x_1 \rangle$$

But $Q_1 |x_1\rangle = 0$ by construction

$$\therefore \hat{K}_1(s) = \frac{1}{\langle x|x\rangle} \frac{1}{s} \langle x_1 | x_1 \rangle \qquad \ldots(5)$$

$\hat{K}_2(s)$ is in the required form

(c) Now $\hat{K}_1(s) = \frac{1}{s\langle x|x\rangle} \langle x_1 | x_1 \rangle = \frac{1}{s\langle x|x\rangle} \langle x | \mathcal{L}Q . Q\mathcal{L} | x \rangle$

$$= \frac{1}{s\langle x|x\rangle} \langle x | \mathcal{L}Q\mathcal{L} | x \rangle \quad \text{(by idempotency of } Q\text{)}$$

$$= \frac{1}{s\langle x|x\rangle} \langle x | \mathcal{L}(1-P)\mathcal{L} | x \rangle$$

$$= \frac{1}{s\langle x|x\rangle} \left[\langle x | \mathcal{L}^2 | x \rangle - \langle x | \mathcal{L} | x \rangle \langle x | x \rangle^{-1} \langle x | \mathcal{L} x \rangle \right]$$

$$= \frac{1}{s\, S_0} \left[S_2 - S_1 \frac{1}{S_0} S_1 \right] = \frac{1}{s} \frac{S_0 S_2 - S_1^2}{S_0^2}$$

which is the required expression.

4a) The required Langevin equation is

$$\zeta \frac{dx(t)}{dt} = F + \theta(t) \qquad \ldots(1)$$

where $\langle \theta(t) \rangle = 0$ and $\langle \theta(t)\theta(t') \rangle = 2\zeta k_B T \, \delta(t-t')$

b) $P(x,t) = \langle \delta(x - x(t)) \rangle$

$$\therefore \frac{\partial P(x,t)}{\partial t} = \left\langle \frac{\partial}{\partial t} \delta(x - x(t)) \right\rangle = -\frac{\partial}{\partial x} \left\langle \delta(x - x(t))\, \dot{x}(t) \right\rangle$$

$$= -\frac{\partial}{\partial x} \left\langle \delta(x - x(t)) \left\{ \frac{F}{\zeta} + \frac{1}{\zeta}\theta(t) \right\} \right\rangle$$

$$= -\frac{F}{\zeta}\frac{\partial}{\partial x} P(x,t) - \frac{1}{\zeta}\frac{\partial}{\partial x} \langle \delta(x - x(t))\, \theta(t) \rangle \qquad \ldots(2)$$

By Novikov's theorem

$$\langle \delta(x - x(t))\, \theta(t) \rangle = \int_0^t dt' \langle \theta(t)\theta(t') \rangle \left\langle \frac{\delta}{\delta\theta(t')} \delta(x - x(t)) \right\rangle$$

$$= -2\zeta k_B T \frac{\partial}{\partial x} \int_0^t dt'\, \delta(t - t') \left\langle \delta(x - x(t)) \frac{\delta x(t)}{\delta\theta(t')} \right\rangle \qquad \ldots(3)$$

From (1), $\frac{d}{dt} \frac{\delta x(t)}{\delta\theta(t')} = \frac{1}{\zeta}\delta(t - t')$

$$\therefore \frac{\delta x(t)}{\delta\theta(t')} = \frac{1}{\zeta}\int_0^t dt''\, \delta(t'' - t') = \frac{1}{\zeta} H(t - t') \qquad \ldots(4)$$

Substituting (4) into (3), and the result into (2), we get

$$\frac{\partial P(x,t)}{\partial t} = -\frac{F}{\zeta}\frac{\partial}{\partial x}P(x,t) - \frac{1}{\zeta}\frac{\partial}{\partial x}(-2\zeta k_B T)\frac{\partial}{\partial x}\int_0^t dt'\,\delta(t - t')\, P(x,t)\,\frac{1}{\zeta}H(t - t')$$

i.e. $$\frac{\partial P(x,t)}{\partial t} = -\frac{F}{\zeta}\frac{\partial}{\partial x}P(x,t) + \frac{k_B T}{\zeta}\frac{\partial^2 P(x,t)}{\partial x^2} \qquad (\text{using } H(0) = \tfrac{1}{2})$$

$$= -\frac{F}{\zeta}\frac{\partial}{\partial x}P(x,t) + D\frac{\partial^2 P(x,t)}{\partial x^2}, \qquad \ldots(5)$$

where $D = k_B T/\zeta$. Eq (5) is the required Fokker-Planck equation.

c) (i) Let $P(x,t) = e^{\alpha x} Q(x,t)$ $\qquad \ldots(6)$

(ii) Substituting (6) into (5) :

$$\frac{\partial P}{\partial t} = e^{\alpha x} \frac{\partial Q}{\partial t}$$

$$\frac{\partial P}{\partial x} = \alpha e^{\alpha x} Q(x,t) + e^{\alpha x} \frac{\partial Q(x,t)}{\partial x}$$

$$= e^{\alpha x}\left(\alpha Q(x,t) + \frac{\partial Q(x,t)}{\partial x}\right)$$

$$\frac{\partial^2 P}{\partial x^2} = e^{\alpha x}\left(\alpha^2 Q(x,t) + 2\alpha \frac{\partial Q(x,t)}{\partial x} + \frac{\partial^2 Q(x,t)}{\partial x^2}\right)$$

$$\therefore e^{\alpha x}\frac{\partial Q}{\partial t} = -\frac{F}{z_1} e^{\alpha x}\left(\alpha Q + \frac{\partial Q}{\partial x}\right) + D e^{\alpha x}\left(\alpha^2 Q + 2\alpha \frac{\partial Q}{\partial x} + \frac{\partial^2 Q}{\partial x^2}\right)$$

$$\therefore \frac{\partial Q}{\partial t} = Q\left(-\frac{\alpha F}{z_1} + \alpha^2 D\right) + \frac{\partial Q}{\partial x}\left(-\frac{F}{z_1} + 2\alpha D\right) + D\frac{\partial^2 Q}{\partial x^2} \qquad \ldots(7)$$

(ii) Choosing α to eliminate $\partial Q/\partial x$ term,

$$-\frac{F}{z_1} + 2\alpha D = 0 \quad \text{or} \quad \alpha = \frac{F}{2Dz_1} \qquad \ldots(8)$$

$$\therefore P(x,t) = e^{Fx/2Dz_1} Q(x,t) \qquad \ldots(9)$$

From (7), with α given by (8),

$$\frac{\partial Q}{\partial t} = Q\left(-\frac{F}{z_1}\cdot\frac{F}{2Dz_1} + D\cdot\frac{F^2}{4D^2z_1^2}\right) + D\frac{\partial^2 Q}{\partial x^2}$$

i.e., $$\frac{\partial Q}{\partial t} = -\frac{F^2}{4Dz_1^2} Q + D\frac{\partial^2 Q}{\partial x^2} \qquad \ldots(10)$$

(iii) Consider a solution of the form $Q(x,t) = T(t)\phi(x)$ ...(11)

Substituting (11) into (10):

$$\phi(x)\frac{\partial T(t)}{\partial t} = -\frac{F^2}{4Dz_1^2} T(t)\phi(x) + T(t)\frac{\partial^2\phi(x)}{\partial x^2}\cdot D \qquad \ldots(12)$$

Divide (12) by $T(t)\phi(x)$,

$$\therefore \quad \frac{1}{T(t)}\frac{\partial T(t)}{\partial t} + \frac{F^2}{4DZ^2} = \frac{D}{\phi(x)}\frac{\partial^2 \phi(x)}{\partial x^2} \qquad \dots(13)$$

Since LHS is independent of x and RHS is independent of t, both sides must equal a constant. Denoting the constant by $-k^2$, Eq. (13) becomes

$$\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = -\left(k^2 + \frac{F^2}{4DZ^2}\right) \qquad \dots(14a)$$

and
$$\frac{D}{\phi(x)}\frac{\partial^2 \phi(x)}{\partial x^2} = -k^2 \qquad \dots(14b)$$

Eq. (14a) is solved as

$$T(t) \propto e^{-(k^2 + F^2/4DZ^2)t} \qquad \dots(15)$$

while (14b) is solved as

$$\phi(x) = A \sin\sqrt{\frac{k^2}{D}}\, x + B \cos\sqrt{\frac{k^2}{D}}\, x \qquad \dots(16)$$

where A and B are integration constants to be determined.
From (9), using (15) and (16),

$$P(x,t) = e^{Fx/2DZ}\, e^{-(k^2 + F^2/4DZ^2)t}\left(A \sin\sqrt{\frac{k^2}{D}}\, x + B \cos\sqrt{\frac{k^2}{D}}\, x\right) \qquad \dots(17)$$

$$\therefore \quad P(0,t) = e^{-(k^2 + F^2/4DZ^2)t}\; B = 0$$

$$\therefore \quad B = 0$$

and $P(x,t)$ becomes

$$P(x,t) = A\, e^{Fx/2DZ}\, e^{-(k^2 + F^2/4DZ^2)t} \sin\sqrt{\frac{k^2}{D}}\, x$$

Also $P(L,t) = 0 = A e^{FL/2DZ}\, e^{-(k^2 + F^2/4DZ^2)t} \sin\sqrt{\frac{k^2}{D}}\, L$

$$\therefore \quad \sqrt{\frac{k^2}{D}}\, L = n\pi, \quad n = 0, 1, 2, \dots \quad \Rightarrow \quad k^2 = \frac{n^2\pi^2 D}{L^2} \qquad \dots(18)$$

$\therefore P(x,t) \propto e^{Fx/2D\zeta}\, e^{-B_n t} \sin\frac{n\pi x}{L}$

where $B_n \equiv \frac{n^2\pi^2 D}{L^2} + \frac{F^2}{4D\zeta^2}$

or $$P(x,t) = \sum_{n=1}^{\infty} C_n \exp\left(\frac{Fx}{2D\zeta} - B_n t\right) \sin\frac{n\pi x}{L} \quad \ldots(19)$$

where C_n is to be determined from the initial condition

(iv) From $P(x,0) = \delta(x-x_0)$, we have

$$\delta(x-x_0) = \sum_{n=1}^{\infty} C_n e^{Fx/2D\zeta} \sin\frac{n\pi x}{L} \quad \ldots(20)$$

Multiply both sides of (20) by $e^{-Fx/2D\zeta}\sin\frac{m\pi x}{L}$, and integrate over x from 0 to L:

$$\int_0^L dx\, e^{-Fx/2D\zeta} \sin\frac{m\pi x}{L}\, \delta(x-x_0) = \sum_{n=1}^{\infty} C_n \int_0^L dx \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}$$

$$\therefore e^{-Fx_0/2D\zeta} \sin\frac{m\pi x_0}{L} = \frac{L}{2}\sum_{n=1}^{\infty} C_n \delta_{n,m} = \frac{L}{2} C_m$$

$$\therefore C_m = \frac{2}{L} e^{-Fx_0/2D\zeta} \sin\frac{m\pi x_0}{L}$$

$$\therefore P(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} e^{\frac{F}{2D\zeta}(x-x_0) - B_n t} \sin\frac{n\pi x_0}{L} \sin\frac{n\pi x}{L} \quad \ldots(20)$$

which is the final expression for $P(x,t)$.

(c) To calculate the first passage time, first calculate the survival probability $S(t;x_0)$, where

$$S(t;x_0) = \int_0^L dx\, P(x,t)$$

$$= \frac{2}{L} e^{-Fx_0/2D\zeta} \sum_{n=1}^{\infty} \int_0^L dx\, e^{Fx/2D\zeta} \sin\frac{n\pi x}{L} \cdot \sin\frac{n\pi x_0}{L}\, e^{-B_n t} \quad \ldots(21)$$

Define $I \equiv \int_0^L dx\, e^{Fx/2D\zeta} \sin\frac{n\pi x}{L}$

Integrate by parts:

$$I = \frac{2D\zeta}{F}\left[e^{Fx/2D\zeta} \sin\frac{n\pi x}{L}\Big|_0^L - \frac{n\pi}{L}\int_0^L dx\, e^{Fx/2D\zeta} \cos\frac{n\pi x}{L} \right]$$

$$= -\frac{2D\zeta}{F}\cdot\frac{n\pi}{L}\cdot\frac{2D\zeta}{F}\left[e^{Fx/2D\zeta}\cos\frac{n\pi x}{L}\Big|_0^L + \frac{n\pi}{L}\cdot I \right]$$

$$= -\frac{n\pi}{L}\left(\frac{2D\zeta}{F}\right)^2\left[(-1)^n e^{FL/2D\zeta} - 1\right] - \left(\frac{2D\zeta}{F}\right)^2\left(\frac{n\pi}{L}\right)^2 I$$

$$\therefore\ I = \frac{(2D\zeta/F)^2 (n\pi/L)\left[e^{FL/2D\zeta}(-1)^{n+1} + 1\right]}{1 + \left(\frac{2D\zeta}{F}\right)^2\left(\frac{n\pi}{L}\right)^2} \quad \ldots(22)$$

From (21) and (22)

$$S(t;x_0) = \frac{2}{L} e^{-Fx_0/2D\zeta}\left(\frac{2D\zeta}{F}\right)^2 \frac{\pi}{L} \sum_{n=1}^{\infty} \frac{e^{-B_n t}\, n \sin(n\pi x_0/L)\left[1 + (-1)^{n+1} e^{FL/2D\zeta}\right]}{1 + \left(\frac{2D\zeta}{F}\right)^2\left(\frac{n\pi}{L}\right)^2} \quad \ldots(23)$$

MFPT is

$$\tau(x_0) = \int_0^{\infty} dt\, S(t;x_0)$$

$$= \frac{2\pi}{L^2} e^{-Fx_0/2D\zeta}\left(\frac{2D\zeta}{F}\right)^2 \sum_{n=1}^{\infty} \frac{n}{B_n} \sin\frac{n\pi x_0}{L}\, \frac{\left[1 + (-1)^{n+1} e^{FL/2D\zeta}\right]}{1 + \left(\frac{2D\zeta}{F}\right)^2\left(\frac{n\pi}{L}\right)^2} \quad \ldots(24)$$

Substituting for B_n and simplifying, (24) becomes

$$\tau(x_0) = \frac{2\pi D}{L^2} \sum_{n=1}^{\infty} \frac{n\left(1+(-1)^{n+1} e^{FL/2D\zeta}\right) e^{-Fx_0/2D\zeta} \sin(n\pi x_0/L)}{\left(\frac{F^2}{4D\zeta^2} + \frac{n^2\pi^2 D}{L^2}\right)^2} \quad \ldots(25)$$

which is the required expression

(c) In the limit $F=0$

$$\tau(x_0) = \frac{2L^2}{\pi^3 D} \sum_{n=1}^{\infty} \frac{1}{n^3}\left(1-(-1)^n\right) \sin\frac{n\pi x_0}{L}$$

$$= \frac{4L^2}{\pi^3 D} \sum_{n=1,3,5\ldots} \frac{1}{n^3} \sin\frac{n\pi x_0}{L} = \frac{4L^2}{\pi^3 D} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x_0/L}{(2n-1)^3} \quad \ldots(26)$$

In the limit $x_0 = L/2$, $\sin(2n-1)\pi x_0/L = \sin(2n-1)\pi/2$
The sum in (26) can now be found from tables

$$\tau = \cancel{\tau(x_0)} = \frac{4L^2}{\pi^3 D} \cdot \frac{\pi^3}{32} = \frac{L^2}{8D} \quad \ldots(27)$$

Plotting $\tau(x_0 = L/2)$ as a function of L and D: