IP326. Problem Set 1. <u>Due Tuesday, February 5, 2019</u>

1. Show that Euler's equation $\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0$ is equivalent to

$$\frac{\partial f}{\partial t} - \frac{d}{dt}\left(f - \dot{x}\frac{\partial f}{\partial \dot{x}} \right) = 0$$

2. Show that if the Lagrangian of a particle is given by

$$L = mc^2\left(1 - \sqrt{1 - \frac{\mathrm{v}^2}{c^2}} \right) - V(\mathbf{r})$$

where m is the mass of the particle, $\mathbf{r}$ its position, $\mathbf{v}$ its velocity, $V(\mathbf{r})$ the potential energy and c the speed of light, then the particle obeys the following relativistic form of Newton's second law:

$$\frac{d}{dt}\left(\frac{m\mathrm{v}_i}{\sqrt{1 - \mathrm{v}^2 / c^2}} \right) = F_i$$

where v_i and F_i are, respectively the *i*th component of the velocity and force.

3. Show that the integral $J = \int_{t_1}^{t_2} dt g(y, \dot{y}, t)$ with $g = y(t)$ has *no* extreme values.

4. Show that if a function f is defined as $f(y, \dot{y}, t) = f_1(y,t) + \dot{y} f_2(y,t)$, then the extremization of the integral

$$I = \int_{t_i}^{t_2} dt\, f(y, \dot{y}, t)$$

leads to the condition $\frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial t} = 0$.

5. If the classical Hamiltonian H does not depend explicitly on time, show that $dH / dt = 0$. What does this mean physically? Does this remain true if H *does* depend explicitly on time?

6. For a particle moving in 3 dimensions under the influence of a spherically symmetric potential $U = U(r)$, write down the Lagrangian and the equations of motion in spherical coordinates (r, θ, ϕ). Show that the Hamiltonian of the system, *H*, is given by $H = K + V$, where *K* is the kinetic energy and *V* the potential energy.

7. For a one-particle system in one dimension, show that the phase space ensemble average of a dynamical variable *B*, defined as $\langle B(t) \rangle = \int d\Gamma f(\Gamma) B(t;\Gamma)$, can also be written as $\langle B(t) \rangle = \int d\Gamma B(\Gamma) f(t;\Gamma)$. (Your proof should be based on the explicit use of the phase space variables *p* and *q*, and not simply on Γ regarded as a single variable, as was done in class.)

8. Suppose that the sign of time is reversed in an *N*-particle system in 3 dimensions at fixed temperature *T* (and at constant *V* and *N*). This corresponds to the phase space transformation

$$\{\mathbf{q}^N, \mathbf{p}^N\} \to \{\mathbf{q}^N, -\mathbf{p}^N\}$$

Assume that the Hamiltonian of this system is a quadratic function of the momenta, and that under time reversal the dynamical variables *A* and *B* transform as $A \to \gamma_A A$ and $B \to \gamma_B B$, where $\gamma_A = \pm 1$ and $\gamma_B = \pm 1$.

(a) Explain why the phase space transformation above is equivalent to time reversal.
(b) Show that the equilibrium ensemble average $\langle A | B \rangle$ vanishes identically if $\gamma_A \neq \gamma_B$.
(c) Show $C_{AB}(t) = \gamma_A \gamma_B C_{AB}(-t) = \gamma_A \gamma_B C^*_{BA}(t)$.

9. Consider an *N*-particle system at constant temperature *T* (and constant *V* and *N*) that has a centre of inversion. Assume that the Hamiltonian of the system is a quadratic function of the momenta. Suppose that under the *parity* operation $\{\mathbf{q}^N, \mathbf{p}^N\} \to \{-\mathbf{q}^N, -\mathbf{p}^N\}$, the dynamical variables *A* and *B* transform as $A \to \gamma_A A$ and $B \to \gamma_B B$, where $\gamma_A = \pm 1$ and $\gamma_B = \pm 1$.

(a) Show that $C_{AB}(t)$ vanishes if *A* and *B* have different parities.
(b) Suppose that under the operation of reflection in the *x*-*z* plane, which corresponds to the transformation $\{x_i, y_i, z_i, p_{ix}, p_{iy}, p_{iz}\} \to \{x_i, -y_i, z_i, p_{ix}, -p_{iy}, p_{iz}\}$, $\forall i \in N$, the variables *A* and *B* transform as $A \to \alpha_A A$ and $B \to \alpha_B B$, where $\alpha_A = \pm 1$ and $\alpha_B = \pm 1$. Show that $C_{AB}(t)$ vanishes if *A* and *B* have different *x*-*z* reflection symmetries. (Assume that the Hamiltonian is invariant under this operation.)