IP326. Lecture 22. Tuesday, March 19, 2019

● Novikov's Theorem

As a prelude to the derivation of an equation for the probability distribution of a dynamical variable from the equation for its time evolution, we will now prove a mathematical result involving the average of a functional of a random variable.

Let $\theta(t)$ denote the random variable in question, with t the time, and $F[\theta]$ some functional of $\theta(t)$. For reasons that will become clear later, we'd like to obtain an expression for $\langle\theta(t)F[\theta]\rangle$, where the angular brackets stand for an average over the distribution of $\theta(t)$. Towards this end, consider first the function $\langle\theta(t)F[\eta+\theta]\rangle$, where $\eta(t)$ is some other arbitrary function of time – independent of $\theta(t)$ – that we'll assume is deterministic. The limit $\eta(t)=0$ recovers the average $\langle\theta(t)F[\theta]\rangle$ that we're interested in.

When $\eta(t)\neq 0$, we can Taylor expand $\langle\theta(t)F[\eta+\theta]\rangle$ around $\theta(t)$. The result is

$$\langle\theta(t)F[\eta+\theta]\rangle = \langle\theta(t)F[\eta]\rangle + \left\langle \theta(t)\int_0^T dt_1 \frac{\delta F[\eta]}{\delta\eta(t_1)}\theta(t_1)\right\rangle + $$

$$+\frac{1}{2!}\left\langle \theta(t)\int_0^T dt_1\int_0^T dt_2 \frac{\delta^2 F[\eta]}{\delta\eta(t_1)\delta\eta(t_2)}\theta(t_1)\theta(t_2)\right\rangle + \cdots \tag{1a}$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^T dt_1\cdots\int_0^T dt_n \frac{\delta^n F[\eta]}{\delta\eta(t_1)\cdots\delta\eta(t_n)}\langle\theta(t_1)\cdots\theta(t_n)\theta(t)\rangle \tag{1b}$$

It is to be understood that the $n=0$ term in Eq. (1b) corresponds to $\langle\theta(t)F[\eta]\rangle$.

Now consider a functional $\chi[J]$, defined as

$$\chi[J] = \left\langle \exp\int_0^T dt\theta(t)J(t)\right\rangle, \tag{2}$$

where the angular brackets denote the same average over the distribution of $\theta(t)$ introduced earlier. Given this expression for $\chi[J]$, averages of the general form $\langle\theta(t_1)\cdots\theta(t_n)\rangle$ can be obtained from the relation

$$\langle \theta(t_1)\cdots\theta(t_n)\rangle = \left.\frac{\delta^n \chi[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right|_{J=0} \tag{3}$$

So Eq. (1b) can be rewritten identically as

$$\langle \theta(t)F[\eta+\theta]\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^T dt_1\cdots\int_0^T dt_n \left.\frac{\delta^n F[\eta]}{\delta\eta(t_1)\cdots\delta\eta(t_n)}\frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}\frac{\delta\chi[J]}{\delta J(t)}\right|_{J=0} \tag{4}$$

The functional derivative $\delta\chi[J]/\delta J(t)$ itself can be rewritten in the form

$$\frac{\delta\chi[J]}{\delta J(t)} = \chi[J]\frac{\delta}{\delta J(t)}\Psi[J] \tag{5a}$$

where

$$\Psi[J] \equiv \ln\,\chi[J] \tag{5b}$$

With these definitions, the *n*th order functional derivative involving *J* in Eq. (4), viz.,

$$I[J] \equiv \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}\frac{\delta\chi[J]}{\delta J(t)} \tag{6a}$$

becomes

$$I[J] \equiv \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}\frac{\delta\Psi[J]}{\delta J(t)}\chi[J], \tag{6b}$$

which can be evaluated using the functional version of Leibnitz's rule for the *n*th derivative of a product of functions. The result is

$$I[J] = \sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\left(\frac{\delta^m}{\delta J(t_1)\cdots\delta J(t_m)}\frac{\delta\Psi[J]}{\delta J(t)}\right)\left(\frac{\delta^{n-m}\delta\chi[J]}{\delta J(t_{m+1})\cdots\delta J(t_n)}\right) \tag{7}$$

where, as before, when $n=0$, the operation of functional differentiation is understood to be omitted. Substituting Eq. (7) into Eq. (4), we get

$$\langle \theta(t)F[\eta+\theta]\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{m!(n-m)!}\int_0^T dt_1\cdots\int_0^T dt_n \frac{\delta^n F[\eta]}{\delta\eta(t_1)\cdots\delta\eta(t_n)}\times$$

$$\times\left.\left(\frac{\delta^m}{\delta J(t_1)\cdots\delta J(t_m)}\frac{\delta\Psi[J]}{\delta J(t)}\right)\left(\frac{\delta^{n-m}\delta\chi[J]}{\delta J(t_{m+1})\cdots\delta J(t_n)}\right)\right|_{J=0} \tag{8}$$

Before proceeding to the next step, it will help to explore the meaning of $\Psi[J]$ further. From its definition [Eq. (5b)], we see that

$$\frac{\delta\Psi[J]}{\delta J(t)} = \frac{1}{\left\langle \exp\int_0^T dt\theta(t)J(t)\right\rangle}\left\langle \theta(t)\int_0^T dt\theta(t)J(t)\right\rangle$$

$$\overset{J\to 0}{=} \left\langle \theta(t)\right\rangle \tag{9a}$$

$$\equiv C_1(t) \tag{9b}$$

In the same way,

$$\frac{\delta\Psi^2[J]}{\delta J(t_1)\delta J(t)} = -\frac{1}{\left\langle \exp\int_0^T dt\theta(t)J(t)\right\rangle^2}\left\langle \theta(t_1)\int_0^T dt\theta(t)J(t)\right\rangle\left\langle \theta(t)\int_0^T dt\theta(t)J(t)\right\rangle +$$

$$+\frac{1}{\left\langle \exp\int_0^T dt\theta(t)J(t)\right\rangle}\left\langle \theta(t)\theta(t_1)\int_0^T dt\theta(t)J(t)\right\rangle$$

$$\overset{J\to 0}{=} \left\langle \theta(t)\theta(t_1)\right\rangle - \left\langle \theta(t)\right\rangle\left\langle \theta(t_1)\right\rangle \tag{10a}$$

$$\equiv C_2(t,t_1) \tag{10b}$$

And similarly

$$\left.\frac{\delta\Psi^3[J]}{\delta J(t_2)\delta J(t_1)\delta J(t)}\right|_{J=0} = \left\langle \theta(t_1)\theta(t_2)\theta(t)\right\rangle - \left\langle \theta(t_1)\theta(t_2)\right\rangle\left\langle \theta(t)\right\rangle - \left\langle \theta(t_2)\theta(t)\right\rangle\left\langle \theta(t_1)\right\rangle$$

$$-\left\langle \theta(t_1)\theta(t)\right\rangle\left\langle \theta(t_2)\right\rangle + 2\left\langle \theta(t_1)\right\rangle\left\langle \theta(t_2)\right\rangle\left\langle \theta(t)\right\rangle \tag{11a}$$

$$\equiv C_3(t,t_1,t_2) \tag{11b}$$

The functions C_1, C_2 and C_3 in Eqs. (9b), (10b) and (11b) define what are referred to as cumulants (the first cumulant being just the mean and the second the variance.) One of the important properties of cumulants is that if the random variable being averaged is Gaussian (meaning it can be defined completely by its mean and variance), then all its cumulants beyond the second are identically 0, a property we shall have occasion to use later. The functional $\Psi[J]$ can therefore be thought of as a cumulant generating functional.

Returning to Eq. (8), we see that

$$\left.\left(\frac{\delta^{n-m}\delta\chi[J]}{\delta J(t_{m+1})\cdots\delta J(t_n)}\right)\right|_{J=0} = \langle \theta(t_{m+1})\cdots\theta(t_n)\rangle \tag{12a}$$

$$\equiv M_{n-m}(t_{m+1},\ldots,t_n) \tag{12b}$$

the M's standing for means. In terms of these means and cumulants, Eq. (8) now becomes

$$\langle \theta(t)F[\eta+\theta]\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{m!(n-m)!}\int_0^T dt_1\cdots\int_0^T dt_n\,\frac{\delta^n F[\eta]}{\delta\eta(t_1)\cdots\delta\eta(t_n)}C_{m+1}(t,t_1,\ldots,t_m)\times$$
$$\times M_{n-m}(t_{m+1},\ldots,t_n) \tag{13}$$

As the next step, we interchange the order of summations, much the way we interchanged orders of integrations. Specifically, we write

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}$$

It can be verified that both ways of writing the summation yield the same results. After this reordering of the sums, if we now introduce a new summation index $k=n-m$, we can rewrite Eq. (13) as

$$\langle \theta(t)F[\eta+\theta]\rangle = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{m!k!}\int_0^T dt_1\cdots\int_0^T dt_{k+m}\,\frac{\delta^{k+m}F[\eta]}{\delta\eta(t_1)\cdots\delta\eta(t_{k+m})}C_{m+1}(t,t_1,\ldots,t_m)M_k(t_{m+1},\ldots,t_{k+m}) \tag{14}$$

This expression can be split up into a product of two factors, one involving integrations up to m, and the other involving integrations from $m+1$to $k+m$. That is,

$$\langle \theta(t)F[\eta+\theta]\rangle = \sum_{m=0}^{\infty}\frac{1}{m!}\int_0^T dt_1\cdots\int_0^T dt_m C_{m+1}(t,t_1,\ldots,t_m)\frac{\delta^m}{\delta\eta(t_1)\cdots\delta\eta(t_m)}\times$$
$$\times\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^T dt_{m+1}\cdots\int_0^T dt_{m+k}M_k(t_{m+1},\ldots,t_{m+k})\frac{\delta^k F[\eta]}{\delta\eta(t_{m+1})\cdots\delta\eta(t_{m+k})} \tag{15}$$

To simplify this expression, let's relabel the integration variables in the second set of integrals as

$$t_{m+1} \to \tau_1,$$

$$t_{m+2} \to \tau_2,$$

$$\vdots$$

$$t_{m+k} \to \tau_k$$

Then Eq. (15) becomes

$$\langle \theta(t) F[\eta+\theta] \rangle = \sum_{m=0}^{\infty} \frac{1}{m!} \int_0^T dt_1 \cdots \int_0^T dt_m C_{m+1}(t, t_1, \ldots, t_m) \frac{\delta^m}{\delta\eta(t_1)\cdots\delta\eta(t_m)} \times$$

$$\times \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^T d\tau_1 \cdots \int_0^T d\tau_k M_k(\tau_1, \ldots, \tau_k) \frac{\delta^k F[\eta]}{\delta\eta(\tau_1)\cdots\delta\eta(\tau_k)} \tag{16}$$

Referring back to Eq. (1b), one can see that the second set of integrals in Eq. (16) is just the definition of $\langle F[\eta+\theta] \rangle$. So this equation reduces to

$$\langle \theta(t) F[\eta+\theta] \rangle = \sum_{m=0}^{\infty} \frac{1}{m!} \int_0^T dt_1 \cdots \int_0^T dt_m C_{m+1}(t, t_1, \ldots, t_m) \frac{\delta^m}{\delta\eta(t_1)\cdots\delta\eta(t_m)} \langle F[\eta+\theta] \rangle \tag{17}$$

Recall that we were actually interested in the $\eta \to 0$ limit of the average $\langle \theta(t) F[\eta+\theta] \rangle$. If we apply this limit to Eq. (17), we'll find that

$$\left. \frac{\delta^m \langle F[\eta+\theta] \rangle}{\delta\eta(t_1)\cdots\delta\eta(t_m)} \right|_{\eta=0} = \left\langle \frac{\delta^m \langle F[\theta] \rangle}{\delta\theta(t_1)\cdots\delta\theta(t_m)} \right\rangle \tag{18}$$

That the RHS of this relation does in fact correctly describe the $\eta \to 0$ limit of $\delta^m \langle F[\eta+\theta] \rangle / \delta\eta(t_1)\cdots\delta\eta(t_m)$ can be illustrated by evaluating this limit for the functional $\langle F[\eta+\theta] \rangle = \left\langle \exp \int_0^T dt A(t)[\eta(t)+\theta(t)] \right\rangle$, where $A(t)$ is some arbitrary function of time. When this expression is functionally differentiated with respect to $\eta(t)$ and then evaluated at $\eta(t)=0$, the result is $\left\langle A(t) \exp \int_0^T dt A(t)\theta(t) \right\rangle$. Exactly the same result is obtained from the expression $\left\langle \frac{\delta}{\delta\theta(t)} \exp \int_0^T dt A(t)\theta(t) \right\rangle$, provided the operation of functional differentiation is carried out *under* the averaging sign.

Using Eq. (18) in Eq. (17), we finally arrive at

$$\left\langle \theta(t)F[\theta]\right\rangle = \sum_{m=0}^{\infty}\frac{1}{m!}\int_0^T dt_1 \cdots \int_0^T dt_m C_{m+1}(t,t_1,\ldots,t_m)\left\langle \frac{\delta^m}{\delta\theta(t_1)\cdots\delta\theta(t_m)}F[\theta]\right\rangle \tag{19}$$

which is Novikov's theorem in its most general form. The theorem is particularly useful when $\theta(t)$ is a Gaussian random variable and $\left\langle \theta(t)\right\rangle = 0$; in that case, as mentioned before, all cumulants beyond the second vanish, so Eq. (19) reduces to

$$\left\langle \theta(t)F[\theta]\right\rangle = \int_0^T dt' \left\langle \theta(t)\theta(t')\right\rangle \left\langle \frac{\delta}{\delta\theta(t')}F[\theta]\right\rangle \tag{20}$$