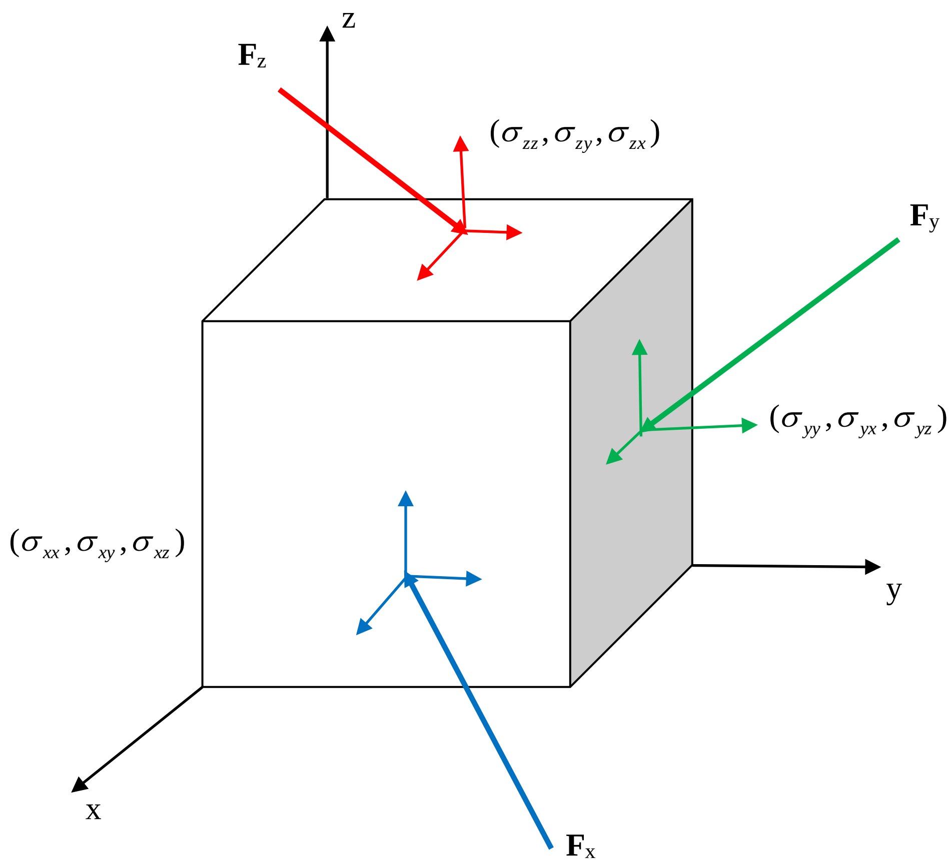
IP326. Lecture 14. Tuesday, Feb. 19, 2019

● Hydrodynamic approach to the shear viscosity (Cont.'d)

We showed earlier that in a given volume of fluid V centered at the point $\mathbf{r}$, the total momentum density $\mathbf{g}(\mathbf{r},t)$ changed with time as a result of two factors: (i) the flow of fluid through the surface $\mathbf{S}$ of V with the velocity $\mathbf{u}(\mathbf{r},t)$ (a process called convection), and (ii) the total force acting on $\mathbf{S}$ from the fluid surrounding it. When these factors were accounted for, we were led to the following equation for the conservation of momentum:

$$\frac{\partial \mathbf{g}(\mathbf{r},t)}{\partial t} = \nabla_{\mathbf{r}} \cdot [\boldsymbol{\sigma} - m\rho(\mathbf{r},t)\mathbf{u}(\mathbf{r},t)\mathbf{u}(\mathbf{r},t)] \qquad (1)$$

where $\rho(\mathbf{r},t)$ is the number density of the fluid, and $\boldsymbol{\sigma}$ is the so-called stress tensor. A given component of $\boldsymbol{\sigma}$, say, σ_{ij}, is to be interpreted as the j component of a force acting on a unit area perpendicular to the i direction. An illustration of what this means is shown below:



The diagonal elements of the stress tensor, viz., σ_{ii}, are called <u>normal stresses</u>, because they represent forces per unit area acting perpendicular or normal to a given plane, while the off-diagonal elements, $\sigma_{ij}, i \neq j$, are called <u>shear stresses</u>, because they represent forces per unit area acting parallel to a given plane.

For a fluid in equilibrium, the shear stresses are 0, while the normal stresses are all equal to each other, and are all independent of position (because otherwise the fluid would move.) So at equilibrium, we have the relation

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -P_0 \tag{2}$$

where P_0 is the equilibrium pressure, and the negative sign indicates that the stresses act outwards, in a direction opposite to P_0, which acts inwards. P_0 is not necessarily the same as the hydrostatic pressure $P(\mathbf{r},t) \equiv P$ at the point **r** at time *t*, which is defined as the normal force per unit area averaged over the three mutually perpendicular planes through the point **r**. That is, $P(\mathbf{r},t) = -(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$.

The stress tenor is also symmetric in its indices. The simple physical reason for this is that if it weren't, the shear stresses σ_{ij} and σ_{ji} would be unbalanced, and they could then potentially generate a torque, causing the fluid to rotate. Since the fluid doesn't rotate, it must be the case that $\sigma_{ij} = \sigma_{ji}$. This allows us to write the components of $\boldsymbol{\sigma}$ as

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij} \tag{3}$$

where τ_{ij} represents the stresses generated when velocity gradients are present, i.e., when there are *viscous forces* in the medium. Because σ_{ij} and δ_{ij} are both symmetric, τ_{ij} must be symmetric too.

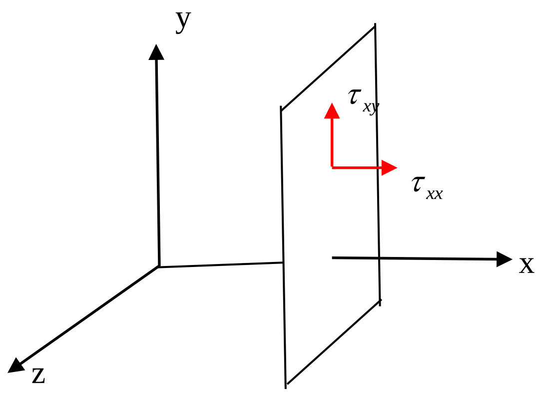
The relation between the stresses τ_{ij} and gradients in the velocity is generally not not known à priori, but when the gradients are small, it's reasonable to assume that they are directly proportional. The most general such linear relationship takes the form

$$\tau_{ij} = \sum_m \sum_n C_{ijmn} \frac{\partial v_m}{\partial q_n}, \quad i, j, m, n = x, y, z \tag{4}$$

where the C_{ijmn} are unknown coefficients (of which, in principle there could be $3^4 = 81$.) But if τ_{ij} is to be symmetric, (i.e., invariant to an interchange of *i* and *j*), then there's only way of combining the terms in (4) that will guarantee this, and that is

$$\tau_{ij} = A\left(\frac{\partial v_i}{\partial q_j} + \frac{\partial v_j}{\partial q_i}\right) + B\left(\frac{\partial v_x}{\partial q_x} + \frac{\partial v_y}{\partial q_y} + \frac{\partial v_z}{\partial q_z}\right)\delta_{ij} \tag{5}$$

where A and B are two other unknown coefficients. We can immediately identify A by considering stress component τ_{xy}, which is the y component of the force per unit area that acts on the surface perpendicular to the x direction (see the figure below):



For this particular component, Eq. (5) reduces to

$$\tau_{xy} = A\frac{\partial v_x}{\partial y} \tag{6}$$

But by definition τ_{xy} is also $-P_{xy}$, the negative of the xy component of the pressure tensor, while P_{xy} itself, as we've seen, is given by $P_{xy} = -\eta\frac{\partial v_x}{\partial y}$. So A in Eq. (5) is nothing but the shear viscosity. That is

$$A = \eta \tag{7}$$

As for B, by general convention, this coefficient is *defined* to be

$$B = -\frac{2}{3}\eta + \kappa \tag{8}$$

where κ is called the <u>dilatational viscosity coefficient</u>. There are reasons for writing B in this form, but we won't get into them, since it will turn out that for the calculation of η, this term in B will eventually prove unimportant.

If we now substitute Eqs. (3), (5), (7) and (8) into Eq. (1), we end up with

$$\frac{\partial(m\rho\mathbf{u})}{\partial t}=-m\nabla\cdot\rho\mathbf{uu}+\nabla\cdot\left[-P\mathbf{1}+\eta\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^{T}\right)-\left(\frac{2}{3}\eta-\kappa\right)\nabla\cdot\mathbf{u1}\right] \tag{9}$$

where the superscript T stands for transpose, and where for convenience the dependence of the density and velocity on **r** and t has been omitted.

Equation (9) is effectively the starting point for our derivation of a formula for η, and to proceed from here, we'll use various vector or tensor identities to simplify each of terms in the equation, starting with the left hand side.

$$\frac{\partial(m\rho\mathbf{u})}{\partial t}=m\left(\mathbf{u}\frac{\partial\rho}{\partial t}+\rho\frac{\partial\mathbf{u}}{\partial t}\right) \tag{10a}$$

$$\nabla\cdot\rho\mathbf{uu}=\sum_{i=1}^{3}\hat{e}_i\frac{\partial}{\partial q_i}\cdot\rho\sum_{j=1}^{3}\sum_{k=1}^{3}\hat{e}_j\hat{e}_k u_j u_k$$

$$=\sum_{i,j,k=1}^{3}\frac{\partial}{\partial q_i}\rho u_j u_k\hat{e}_i\cdot\hat{e}_j\hat{e}_k=\sum_{i,j,k=1}^{3}\frac{\partial}{\partial q_i}\rho u_j u_k\delta_{ij}\hat{e}_k$$

$$=\sum_{i,k=1}^{3}\frac{\partial}{\partial q_i}\rho u_i u_k\hat{e}_k$$

$$=\sum_{i,k=1}^{3}\left(\hat{e}_k u_k\frac{\partial}{\partial q_i}\rho u_i+\rho u_i\frac{\partial}{\partial q_i}\hat{e}_k u_k\right)$$

$$=\mathbf{u}\nabla\cdot\rho\mathbf{u}+\rho\,\mathbf{u}\cdot\nabla\mathbf{u} \tag{10b}$$

$$\nabla\cdot P\mathbf{1}=\sum_{i=1}^{3}\hat{e}_i\frac{\partial}{\partial q_i}\cdot P\sum_{j,k=1}^{3}\hat{e}_j\hat{e}_k\delta_{jk}$$

$$=\sum_{i,j=1}^{3}\frac{\partial}{\partial q_i}P\hat{e}_i\cdot\hat{e}_j\hat{e}_j=\sum_{i,j=1}^{3}\frac{\partial}{\partial q_i}P\delta_{ij}\hat{e}_j=\sum_{i=1}^{3}\hat{e}_i\frac{\partial}{\partial q_i}P$$

$$=\nabla P \tag{10c}$$

$$\nabla\cdot\nabla\mathbf{u}=\sum_{i=1}^{3}\hat{e}_i\frac{\partial}{\partial q_i}\cdot\sum_{j=1}^{3}\hat{e}_j\frac{\partial}{\partial q_j}\sum_{k=1}^{3}\hat{e}_k u_k$$

$$= \sum_{i,j,k=1}^{3} \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} u_k \hat{e}_i \cdot \hat{e}_j \hat{e}_k = \sum_{i,j,k=1}^{3} \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} u_k \delta_{ij} \hat{e}_k$$

$$= \sum_{i,k=1}^{3} \frac{\partial^2}{\partial q_i^2} u_k \hat{e}_k$$

$$= \nabla^2 \mathbf{u} \tag{10d}$$

$$\nabla \cdot (\nabla \mathbf{u})^T = \sum_{i=1}^{3} \hat{e}_i \frac{\partial}{\partial q_i} \cdot \sum_{j,k=1}^{3} \hat{e}_j \hat{e}_k \frac{\partial}{\partial q_k} u_j$$

$$= \sum_{i,j,k=1}^{3} \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_k} u_j \hat{e}_i \cdot \hat{e}_j \hat{e}_k = \sum_{i,j,k=1}^{3} \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_k} u_j \delta_{ij} \hat{e}_k$$

$$= \sum_{i,k=1}^{3} \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_k} u_i \hat{e}_k = \sum_{i,k=1}^{3} \hat{e}_k \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_i} u_i$$

$$= \nabla \nabla \cdot \mathbf{u} \tag{10e}$$

$$\nabla \cdot \nabla \cdot \mathbf{u} \mathbf{1} = \sum_{i=1}^{3} \hat{e}_i \frac{\partial}{\partial q_i} \cdot \sum_{j=1}^{3} \frac{\partial}{\partial q_j} u_j \sum_{k=1}^{3} \hat{e}_k \hat{e}_k$$

$$= \sum_{i,j,k=1}^{3} \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} u_j \hat{e}_i \cdot \hat{e}_k \hat{e}_k = \sum_{i,j,k=1}^{3} \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} u_j \delta_{ik} \hat{e}_k$$

$$= \sum_{i,j}^{3} \hat{e}_i \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} u_j$$

$$= \nabla \nabla \cdot \mathbf{u} \tag{10f}$$

Substituting Eqs. (10a)-(10f) into (9), we get

$$m\mathbf{u} \frac{\partial \rho}{\partial t} + m\rho \frac{\partial \mathbf{u}}{\partial t} = -m\mathbf{u} \nabla \cdot \rho \mathbf{u} - m\rho \mathbf{u} \cdot \nabla \mathbf{u} - \nabla P + \eta (\nabla^2 \mathbf{u} + \nabla \nabla \cdot \mathbf{u}) - (\frac{2\eta}{3} - \kappa) \nabla \nabla \cdot \mathbf{u} \tag{11}$$

This can be simplified by recalling the equation for the conservation of mass: $m\partial \rho / \partial t = -m \nabla \cdot \rho \mathbf{u}$. The substitution of this relation into Eq. (11) leads to

$$m\rho\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)\mathbf{u}=-\nabla P+\eta\nabla^2\mathbf{u}+\left(\frac{1}{3}\eta+\kappa\right)\nabla\nabla\cdot\mathbf{u} \tag{12}$$

This is the Navier-Stokes equation, and it is the key hydrodynamic equation that governs the flow of a viscous fluid.